Addition sequences and numerical evaluation of modular forms

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Modular forms

A **modular form** of weight k is a holomorphic function on $\mathbb{H} = \{\tau : \tau \in \mathbb{C}, \mathrm{im}(\tau) > 0\}$ satisfying

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

for any integers a, b, c, d with $ad - bc = 1$. A **modular function** is meromorphic and has weight $k = 0$.

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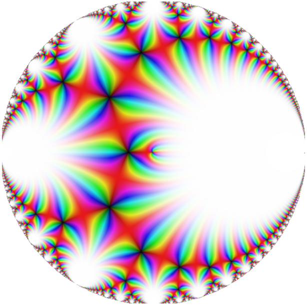
Since $f(\tau) = f(\tau + 1)$, it has a Fourier series (q-expansion)

$$f(\tau) = \sum_{n=-m}^{\infty} c_n e^{2i\pi n\tau} = \sum_{n=-m}^{\infty} c_n q^n$$

where $q = e^{2\pi i\tau}$ (note that $|q| < 1$).

Picture of a modular function: the j-function $j(\tau)$

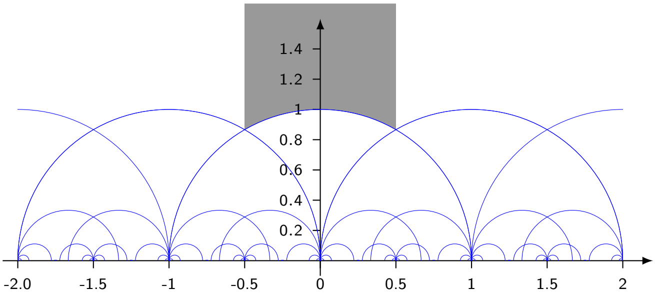


As a function of $\tau \in [-2,2] + [0,1]i$ (top) and of q (bottom).

Numerical evaluation

By repeated use of $\tau \to \tau + 1$ or $\tau \to -1/\tau$, we can move τ to the *fundamental domain* $\left\{\tau \in \mathbb{H} : |z| \geq 1, |\mathrm{Re}(z)| \leq \frac{1}{2}\right\}$.

In the fundamental domain, $|q| \leq \exp(-\pi\sqrt{3}) = 0.00433\ldots$, which gives rapid convergence of the q-expansion.



[Source for illustration: user "Tom Bombadil" on TeX StackExchange.]

Example: the j-function

Any elliptic curve $y^2 = x^3 + ax + b$ over $\mathbb{C}$ can be identified with a complex lattice $(1, \tau)$.

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The j-function describes isomorphism classes of elliptic curves.
It is a modular function ($j(\tau) = j(\tau + 1) = j(-1/\tau)$) and has the q-expansion

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

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The j-function has magical properties:

- At certain algebraic τ, the value $j(\tau)$ is also algebraic
- $e^{\pi\sqrt{163}} = 640320^3 + 743.99999999999925007\ldots$
- The q-expansion is related to the "monster group"
- ...

The Hilbert class polynomial

For $D < 0$ congruent to 0 or 1 mod 4,

$$H_D(x) = \prod_{(a,b,c)} \left(x - j\left(\frac{-b + \sqrt{D}}{2a} \right) \right) \in \mathbb{Z}[x]$$

where (a, b, c) is taken over all the primitive reduced binary quadratic forms $ax^2 + bxy + cy^2$ with $b^2 - 4ac = D$.

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Application: constructing elliptic curves with a prescribed number of points over a finite field (useful in primality proving, cryptography).

The first few Hilbert class polynomials

D	H_D
-3	x
-4	$x - 1728$
-7	$x + 3375$
-8	$x - 8000$
-11	$x + 32768$
-12	$x - 54000$
-15	$x^2 + 191025x - 121287375$
-16	$x - 287496$
-19	$x + 884736$
-20	$x^2 - 1264000x - 681472000$
-23	$x^3 + 3491750x^2 - 5151296875x + 12771880859375$
-24	$x^2 - 4834944x + 14670139392$
-27	$x + 12288000$
-28	$x - 16581375$
-31	$x^3 + 39491307x^2 - 58682638134x + 1566028350940383$

Numerical example

The quadratic forms with discriminant $D = -31$ are

$$x^2 + xy + 8y^2, \quad 2x^2 + xy + 4y^2, \quad 2x^2 - xy + 4y^2$$

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Therefore $H_{-31} = (x - j_1)(x - j_2)(x - j_3)$ where

$$j_1 = j\left(\tfrac{-1+\sqrt{-31}}{2}\right), \quad j_2 = j\left(\tfrac{-1+\sqrt{-31}}{4}\right), \quad j_3 = \bar{j_2} = j\left(\tfrac{+1+\sqrt{-31}}{4}\right)$$

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Using ball arithmetic with 73 bits of precision, we compute

$$\begin{aligned} j_1 &= [-39492793.91155624414 \pm 6.10 \cdot 10^{-12}] \\ j_2 &= [743.455778122071940 \pm 3.22 \cdot 10^{-16}] \\ &\quad + [6253.062846903285089 \pm 8.87 \cdot 10^{-16}]i \end{aligned}$$

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Expanding gives $H_{-31} = x^3 + c_2x^2 + c_1x + c_0$ where

$$\begin{aligned} c_2 &= [39491307.00000000000 \pm 2.44 \cdot 10^{-12}] \\ c_1 &= [-58682638134.0000000 \pm 1.61 \cdot 10^{-8}] \\ c_0 &= [1566028350940383.000 \pm 3.22 \cdot 10^{-4}] \end{aligned}$$

Computing H_D

Problem: for large D, H_D is huge!

- $\deg(H_D) = O(|D|^{1/2+\varepsilon})$
- $\max \log_2 |[x^k]H_D| = O(|D|^{1/2+\varepsilon})$
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Enge (2009): complexity analysis and tight coefficient bounds for the complex analytic method. Depends on asymptotically fast polynomial arithmetic over $\mathbb{R}$. Without complete proofs for floating-point rounding errors.

New: a fast, rigorous implementation

Sage: complex analytic (floating-point)

Pari/GP: CRT method, by Hamish Ivey-Law

CM: complex analytic (floating-point), by Andreas Enge

Arb: complex analytic (ball arithmetic), by FJ

D	deg	bits	Sage	Pari/GP	CM	Arb
$-1\,000\,003$	105	8527	2.1 s	12 s	0.7 s	0.33 s
$-10\,000\,003$	706	50889	601 s	290 s	101 s	46 s
$-100\,000\,003$	1702	153095			1822 s	750 s

The expensive steps when computing H_D

A. Compute numerical approximations of the $j(\tau)$ values

B. Multiply together the linear factors $(x - j(\tau))$

In practice, the bottleneck is A.

This leads to the question of how much this task (and more generally, numerical evaluation of other modular forms/functions) can be optimized.

Choice of q-expansion

Recall

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

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In practice, one rewrites the modular form/function one wishes to compute in terms of either:

- The Dedekind eta function
- Jacobi theta functions

These functions not only have small and explicit coefficients ($c_n = \pm 1$), but their q-expansions are *sparse*.

The Dedekind eta function

$$\begin{aligned}\eta(\tau) &= e^{\pi i\tau/12}\prod_{k=1}^{\infty}(1-q^k) = e^{\pi i\tau/12}\sum_{k=-\infty}^{\infty}(-1)^k q^{(3k^2-k)/2}\\ &= e^{\pi i\tau/12}\left(1-q-q^2+q^5+q^7-q^{12}-q^{15}+\dots\right)\end{aligned}$$

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$$= e^{\pi i \tau/12} \left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \ldots\right)$$

The exponents $P(k) = (3k^2 - k)/2$ are the *pentagonal numbers*.

$$P(0), P(1), P(2), \ldots = 0, 1, 5, 12, 22, \ldots$$

$$P(-1), P(-2), \ldots = 2, 7, 15, 26, \ldots$$

For d digits, we only need $O(d^{1/2})$ terms of the q-expansion!

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$$j(\tau) = \left(\left(\frac{\eta(\tau)}{\eta(2\tau)} \right)^8 + 2^8 \left(\frac{\eta(2\tau)}{\eta(\tau)} \right)^{16} \right)^3$$

Properties of the eta function

It is a modular form of weight $1/2$:

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)=\varepsilon(a,b,c,d)(c\tau+d)^{1/2}\eta(\tau)$$

where $\varepsilon(a,b,c,d)$ is a certain 24th root of unity.

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It generates the partition function $p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}$:

$$e^{\pi i\tau/12}/\eta(\tau) = \sum_{n=0}^{\infty} p(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \dots$$

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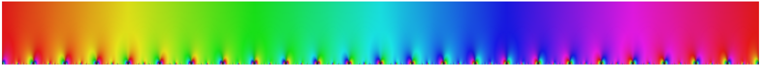
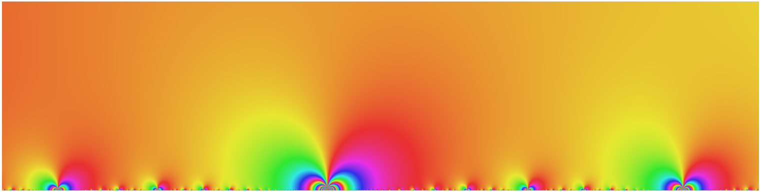
It has interesting special values such as

$$\eta(i) = \frac{\Gamma\left(1/4\right)}{2\pi^{3/4}}$$

Pictures of $\eta(\tau)$



Overview: $\tau \in [0, 24] + [0, 1]i$



Deep zoom: $\tau \in [\sqrt{2}, \sqrt{2} + 10^{-101}] + [0, 2.5 \times 10^{-102}]i$

Computing the Dedekind eta function

$$\eta(\tau) = e^{\pi i \tau / 12} \left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \ldots\right)$$

We compute $\eta(i)$ to 50-digit precision.

$+q^0$	$+1.0000000000000000000000000000000000000000000000000$
$-q^1$	$-0.0018674427317079888144302129348270303934228050024\ldots$
$-q^2$	$-0.0000034873423562089954917752662652081277882033550\ldots$
$+q^5$	$+0.0000000000000227110106832409383867927523909354754\ldots$
$+q^7$	$+0.0000000000000000000792010695079811226359059422276\ldots$
$-q^{12}$	$-0.0000000000000000000000000000000017987363357198674\ldots$
$-q^{15}$	$-0.0000000000000000000000000000000000000000117141123\ldots$

$$\eta(i) = 0.76822542232605665900259417957618064451786691446480\ldots$$

Jacobi theta functions

$$\theta_1(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i[(n+\frac{1}{2})^2\tau+(2n+1)z+n-\frac{1}{2}]} = 2q' \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)} \sin((2n+1)\pi z)$$

$$\theta_2(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i[(n+\frac{1}{2})^2\tau+(2n+1)z]} = 2q' \sum_{n=0}^{\infty} q^{n(n+1)} \cos((2n+1)\pi z)$$

$$\theta_3(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i[n^2\tau+2nz]} = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos(2n\pi z)$$

$$\theta_4(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i[n^2\tau+2nz+n]} = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos(2n\pi z)$$

$$q = \exp(\pi i\tau), \qquad q' = \exp(\pi i\tau/4)$$

We only require the "theta constants" which have $z = 0$.

Theta constants

$$\theta_2(\tau) = 2q' \sum_{n=0}^{\infty} q^{n(n+1)} = 2q'(1 + q^2 + q^6 + q^{12} + q^{20} + \dots)$$

$$\theta_3(\tau) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \dots$$

$$\theta_4(\tau) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} = 1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \dots$$

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$$j(\tau) = 32 \frac{(\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8)^3}{(\theta_2(\tau)\theta_3(\tau)\theta_4(\tau))^8}$$

$$2\eta(\tau)^3 = \theta_2(\tau)\theta_3(\tau)\theta_4(\tau)$$

Addition sequences

We call a sequence of increasing positive integers $c_0, c_1, \dots$ an *addition sequence* if for every $c_k \neq 1$, there exist $i, j < k$ such that

$$c_k = c_i + c_j.$$

More formally, an addition sequence specifies the triples (c_k, c_i, c_j).

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An addition sequence of length n for a finite list of exponents $c_0, c_1, \ldots, c_n$ gives us an algorithm to compute the powers

$$q^{c_0}, q^{c_1}, q^{c_2}, \ldots, q^{c_n}$$

using n multiplications

$$q^{c_k} = q^{c_i} \cdot q^{c_j}.$$

Examples

Some sequences are already addition sequences:

$$n = 1, 2, 3, 4, 5, 6, \ldots$$

$$10n = (1, 2, 4, 5), 10, 20, 30, 40, 50, \ldots$$

$$F_n = 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

$$2^n = 1, 2, 4, 8, 16, 32, 64, 128, 256, \ldots$$

Others are not, and have to be extended:

$$n(n+1) = 2, 6, 12, 20, 30, 42, \ldots$$

$$n^2 : 1, 4, 9, 16, 25, 36, 49, 64, \ldots$$

$$3n(n-1)/2 = 1, 2, 5, 7, 12, 15, 22, 26, \ldots$$

$$3^n = 1, 3, 9, 27, 81, 243, 729, 2187, \ldots$$

Side note

Downey, Leong, Sethi (1981): the associated decision problem

> *Given $c_0, \ldots, c_n$ and a bound N, is there an addition sequence for $c_0, \ldots, c_n$ of length $\leq N$?*

is NP-complete.

A general way to construct addition sequences

Given a set of positive integers $C = \{c_0, \ldots, c_n\}$ with $c_0 < c_1 < \ldots < c_n$, it is clearly possible to construct an addition sequence for C having length

$$O(n \log c_n).$$

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A more elaborate method gives (Yao 1976, cited in Knuth 4.6.3 exercise 37)

$$O\left(\log c_n + n \frac{\log c_n}{\log \log c_n} + \frac{\log c_n \log \log \log c_n}{(\log \log c_n)^2}\right)$$

Shorter addition sequences for polynomials

For any integer-valued polynomial $f \in \mathbb{Q}[X]$ of degree D, the consecutive values $f(1), f(2), \ldots, f(n)$ can be computed using $Dn + O(1)$ additions.

Use the system of recurrences given by iterated differences:

$$\begin{aligned} f(X) = f_D(X) &= f_D(X-1) + f_{D-1}(X-1) \\ f_{D-1}(X) &= f_{D-1}(X-1) + f_{D-2}(X-1) \\ \vdots &= \vdots \\ f_1(X) &= f_1(X-1) + f_0(X-1) \\ f_0(X) &= \text{constant} \end{aligned}$$

For $D = 2$ (including trigonal, square, pentagonal numbers), this method gives an addition sequence of length $2n + O(1)$.

Even shorter addition sequences for polynomials

Dobkin, Lipton (1980):

1. The n first squares $c_n = n^2$ can be computed using

$$n + O(n/\sqrt{\log n}) = n + o(n)$$

additions.

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2. For the squares, cubes, ..., and more generally k-th powers $c_n = n^k$, evaluating the first n terms requires at least $n + n^{2/3-\varepsilon}$ additions. This result also holds for a larger class of polynomials.

New results (EHJ)

For trigonal, square and pentagonal numbers:

1. Theorems regarding addition sequences of special form. The special addition sequences allow computing $\sum_{k=0}^{n} q^{c_k}$ using $n + o(n)$ multiplications (heuristically).

2. Computing $\sum_{k=0}^{n} q^{c_k}$ using $o(n)$ multiplications.

Pentagonal numbers

Theorem $c = 2a + b$**:** Every pentagonal number $c \geq 2$ is the sum of a smaller one and twice a smaller one, that is, there are pentagonal numbers $a, b < c$ such that $c = 2a + b$.

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Conjecture: the first n pentagonal numbers can be computed using $n + O(n/\log n)$ additions

Pentagonal numbers

c	$a+b$	$2a+b$
2	$(1,1)$	$(1,0)$
5		$(2,1)$
7	$(2,5)$	$(1,5)$
12	$(5,7)$	$(5,2)$
15		$(5,5)$ $(7,1)$
22	$(7,15)$	$(5,12)$
26		$(2,22)$ $(7,12)$ $(12,2)$
35		$(15,5)$
40	$(5,35)$	$(7,26)$
51		$(22,7)$
57	$(22,35)$	$(26,5)$
70	$(35,35)$	$(15,40)$ $(22,26)$ $(35,0)$
77	$(7,70)$ $(26,51)$	$(35,7)$
92	$(15,77)$ $(22,70)$ $(35,57)$	$(26,40)$ $(35,22)$ $(40,12)$
100		$(15,70)$

Squares and trigonal numbers

We want to compute $\theta_2(\tau)$, $\theta_3(\tau)$ and $\theta_4(\tau)$ simultaneously.

The *quarter-squares* $t(n) = \lfloor (n+1)^2/4 \rfloor$ consist of the squares $t(2m-1) = m^2$ and trigonal numbers $t(2m) = m(m+1)$ interleaved in increasing order.

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Theorem $c = 2a + b$**:** Every quarter-square $c \geq 2$ is the sum of a smaller one and twice a smaller one, that is, there are quarter-squares $a, b < c$ such that $c = 2a + b$.

$\Rightarrow q^c = (q^a)^2 \cdot q^b$ always works

Quarter-squares

c	$a+b$	$2a+b$
2	$(1,1)$	$(1,0)$
4	$(2,2)$	$(1,2)$ $(2,0)$
6	$(2,4)$	$(1,4)$ $(2,2)$
9		$(4,1)$
12	$(6,6)$	$(4,4)$ $(6,0)$
16	$(4,12)$	$(2,12)$ $(6,4)$
20	$(4,16)$	$(2,16)$ $(4,12)$ $(9,2)$
25	$(9,16)$	$(12,1)$
30		$(9,12)$ $(12,6)$
36	$(6,30)$ $(16,20)$	$(12,12)$ $(16,4)$
42	$(6,36)$ $(12,30)$	$(6,30)$ $(20,2)$
49		$(12,25)$ $(20,9)$
56	$(20,36)$	$(20,16)$ $(25,6)$
64		$(4,56)$ $(30,4)$
72	$(16,56)$ $(30,42)$ $(36,36)$	$(4,64)$ $(30,12)$ $(36,0)$
81	$(9,72)$ $(25,56)$	$(16,49)$ $(36,9)$
90	$(9,81)$	$(9,72)$ $(30,30)$ $(42,6)$
100	$(36,64)$	$(42,16)$ $(49,2)$

Proof of $c = 2a + b$ for quarter-squares

If $t(n) = \lfloor (n+1)^2/4 \rfloor$,

$$\begin{aligned}
t(6n+0) &= 2t(4n) + t(2n-2)\\
t(6n+1) &= 2t(4n) + t(2n+1)\\
t(6n+2) &= 2t(4n+1) + t(2n)\\
t(6n+3) &= 2t(4n+2) + t(2n-1)\\
t(6n+4) &= 2t(4n+2) + t(2n+2)\\
t(6n+5) &= 2t(4n+3) + t(2n+1).
\end{aligned}$$

Proof of $c = 2a + b$ for pentagonal numbers (sketch)

The increasing map

$$c \to \sqrt{24c+1}.$$

is a bijection between the pentagonal numbers $P(n) = (3n^2 - n)/2, n \in \mathbb{Z}$ and the positive integers coprime to 6.

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The existence of a solution $c = 2a + b$ is equivalent to: for $z \geq 11$ coprime to 6, there are positive x and y coprime to 6 such that

$$z^2 + 2 = 2x^2 + y^2$$

other than the trivial solution $(x, y) = (1, z)$.

Proof of $c = 2a + b$ for pentagonal numbers (sketch)

Solutions of $k = 2x^2 + y^2$ correspond to elements $x\sqrt{-2} + y$ with norm k in the ring of integers $\mathbb{Z}[\sqrt{-2}]$ of $\mathbb{Q}(\sqrt{-2})$. Standard methods allow counting solutions via the prime factorization of k.

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We can show that if $k = z^2 + 2$ has at least two distinct prime factors, there must be at least two solutions (with x, y positive and coprime to 6): the trivial $(x, y) = (1, z)$, and at least one that is nontrivial.

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Note that k is always divisible by 3. Therefore, a second solution is guaranteed to exist unless k is a power of 3 (other than $k = 3$ and $k = 27$).

Proof of $c = 2a + b$ for pentagonal numbers (sketch)

Proposition: The only solutions of $3^n = x^2 + 2$ with $x, n \geq 0$ are $(n, x) = (1, 1)$ and $(3, 5)$.

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After ruling out various cases, this becomes

$$-2 = x^2 - 243y^2$$

This Pell-type equation can be solved explicitly. All the solutions (up to signs) are given by $x_0 = 265, y_0 = 17$, and for $k \geq 1$,

$$\begin{aligned} x_k &= 70226x_{k-1} + 1094715y_{k-1} \\ y_k &= 4505x_{k-1} + 70226y_{k-1} \end{aligned}$$

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One sees that every y_k is divisible by 17, and therefore cannot be a power of 3.

Alternative proof

Hirschhorn (2009) shows that the number of ways an integer c can be written as $2a + b$ with a, b pentagonal numbers is

$$d_{1,8}(24c+3) - d_{7,8}(24c+3) - \big(d_{1,8}((8c+1)/3) - d_{7,8}((8c+1)/3)\big),$$

where $d_{i,j}$ counts the number of positive divisors that are $i \pmod{j}$ for integral arguments, and equals 0 for non-integral rational arguments.

That this is ≥ 1 when c is pentagonal and $a, b < c$ can be shown using quadratic reciprocity and the proposition on the previous slide (we still need a similar amount of calculations).

Using less than n multiplications

Theorem: For any integer-valued quadratic polynomial $F(X)$,

$$\sum_{i=0}^{n} q^{F(i)}$$

can be computed using

$$O(n/\log^r n)$$

multiplications, for any $r > 0$.

Rectangular splitting

Paterson-Stockmeyer, 1973: method for evaluating *dense* series:

$$\sum_{k=0}^{N} \square q^k =$$

$$\begin{aligned} & (\square + \square q + \square q^2 + \ldots + \square q^{m-1}) \\ +q^m & (\square + \square q + \square q^2 + \ldots + \square q^{m-1}) \\ +q^{2m} & (\square + \square q + \square q^2 + \ldots + \square q^{m-1}) \\ +q^{3m} & (\square + \square q + \square q^2 + \ldots + \square q^{m-1}) \\ & \vdots \end{aligned}$$

Cost is $m + N/m$ multiplications, or $O(N^{1/2})$ with $m \sim N^{1/2}$.

No improvement for our sparse series with $n = O(N^{1/2})$ terms.

Rectangular splitting

Idea: choose m such that $F(X)$ takes few distinct values mod m.

Consider $F(X) = X^2$ and

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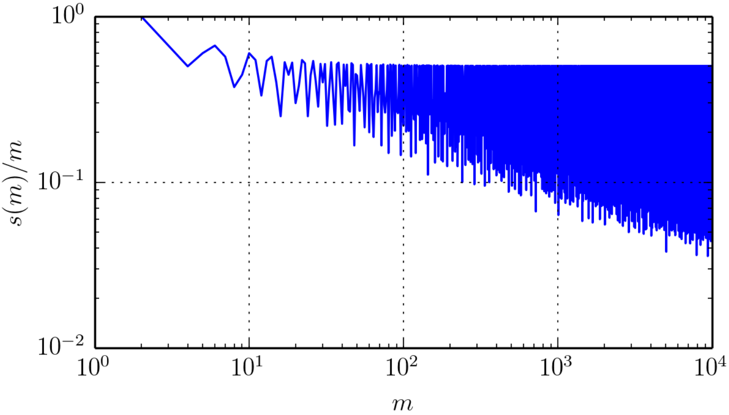
We need $O(s(m) \log m + N/m)$ multiplications, where we want m large and $s(m)$ small.

This suggests looking for m such that

$$\frac{s(m)}{m}$$

is small.

Successive minima



The m such that $s(m)/m < s(m')/m'$ for all $m' < m$ are a good choice.

k	$m = \text{A085635}(k)$	$s(m) = \text{A084848}(k)$	$s(m)/m$
1	$2 = 2$	2	1.0
2	$3 = 3$	2	0.67
3	$4 = 2^2$	2	0.50
4	$8 = 2^3$	3	0.38
5	$12 = 2^2 \cdot 3$	4	0.33
6	$16 = 2^4$	4	0.25
7	$32 = 2^5$	7	0.22
8	$48 = 2^4 \cdot 3$	8	0.17
9	$80 = 2^4 \cdot 5$	12	0.15
10	$96 = 2^5 \cdot 3$	14	0.15
11	$112 = 2^4 \cdot 7$	16	0.14
12	$144 = 2^4 \cdot 3^2$	16	0.11
13	$240 = 2^4 \cdot 3 \cdot 5$	24	0.10
14	$288 = 2^5 \cdot 3^2$	28	0.097
15	$336 = 2^4 \cdot 3 \cdot 7$	32	0.095
16	$480 = 2^5 \cdot 3 \cdot 5$	42	0.088

k	m	$s(m)$	$s(m)/m$
17	$560 = 2^4 \cdot 5 \cdot 7$	48	0.086
18	$576 = 2^6 \cdot 3^2$	48	0.083
19	$720 = 2^4 \cdot 3^2 \cdot 5$	48	0.067
20	$1008 = 2^4 \cdot 3^2 \cdot 7$	64	0.063
21	$1440 = 2^5 \cdot 3^2 \cdot 5$	84	0.058
22	$1680 = 2^4 \cdot 3 \cdot 5 \cdot 7$	96	0.057
23	$2016 = 2^5 \cdot 3^2 \cdot 7$	112	0.056
24	$2640 = 2^4 \cdot 3 \cdot 5 \cdot 11$	144	0.055
25	$2880 = 2^6 \cdot 3^2 \cdot 5$	144	0.050
26	$3600 = 2^4 \cdot 3^2 \cdot 5^2$	176	0.049
27	$4032 = 2^6 \cdot 3^2 \cdot 7$	192	0.048
28	$5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7$	192	0.038
	$\vdots$		
94	$41801760 = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 29$	211680	0.0051
95	$42325920 = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19$	211680	0.0050
96	$48454560 = 2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 19 \cdot 23$	241920	0.0050
97	$49008960 = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$	217728	0.0044
98	$54774720 = 2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 19$	241920	0.0044
99	$61261200 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17$	266112	0.0043
100	$68468400 = 2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 19$	295680	0.0043

The function $s(m)$ is multiplicative, and takes the values

$$s(m) = \begin{cases} \frac{1}{2}p^e - \frac{1}{2}p^{e-1} + \frac{p^{e-1} - p^{(e+1) \bmod 2}}{2(p+1)} + 1 & \text{for } p \text{ odd;} \\ 2 & \text{for } p = 2 \text{ and } e \leq 2; \\ 2^{e-3} + \frac{2^{e-3} - 2^{(e+1) \bmod 2}}{3} + 2 & \text{for } p = 2 \text{ and } e \geq 3, \end{cases}$$

at prime powers $m = p^e$.

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The construction is analogous for other quadratic polynomials.

Successive minima for trigonal numbers

k	m	$t(m)$	$t(m)/m$
1	$2 = 2$	1	0.50
2	$6 = 2 \cdot 3$	2	0.33
3	$10 = 2 \cdot 5$	3	0.30
4	$14 = 2 \cdot 7$	4	0.29
5	$18 = 2 \cdot 3^2$	4	0.22
6	$30 = 2 \cdot 3 \cdot 5$	6	0.20
7	$42 = 2 \cdot 3 \cdot 7$	8	0.19
8	$66 = 2 \cdot 3 \cdot 11$	12	0.18
9	$70 = 2 \cdot 5 \cdot 7$	12	0.17
10	$90 = 2 \cdot 3^2 \cdot 5$	12	0.13
	$\vdots$		
100	$25160850 = 2 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23$	199584	0.0079
101	$25675650 = 2 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 19$	203280	0.0079
102	$28120950 = 2 \cdot 3^2 \cdot 5^2 \cdot 11 \cdot 13 \cdot 19 \cdot 23$	221760	0.0079
103	$29099070 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	181440	0.0062

Successive minima for pentagonal numbers

k	m	$p(m)$	$p(m)/m$
1	$2 = 2$	2	1.0
2	$5 = 5$	3	0.60
3	$7 = 7$	4	0.57
4	$11 = 11$	6	0.55
5	$13 = 13$	7	0.54
6	$17 = 17$	9	0.53
7	$19 = 19$	10	0.53
8	$23 = 23$	12	0.52
9	$25 = 5^2$	11	0.44
10	$35 = 5 \cdot 7$	12	0.34
	$\vdots$		
100	$4555915 = 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19 \cdot 31$	120960	0.027
101	$5159245 = 5 \cdot 7 \cdot 13 \cdot 17 \cdot 23 \cdot 29$	136080	0.026
102	$5311735 = 5 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	136080	0.026
103	$6697405 = 5 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 29$	170100	0.025

Example: computing θ_3

Suppose we want to compute

$$1 + 2\sum_{k=1}^{n} q^{k^2} \approx 1 + \sum_{k=1}^{\infty} 2q^{k^2}$$

for $q = \exp(-\pi)$, with n such that the error is less than 2^{-B}

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for $q = \exp(-\pi)$, with n such that the error is less than 2^{-B}

B	n	$\#(n^2)$	m	$s(m)$	$\#(\mathrm{mod}\ m)$	$\#(\mathrm{tot})$	Speedup
10^3	14	23	48	8	12	16	1.44
10^4	46	71	144	16	23	37	1.92
10^5	148	228	720	48	57	87	2.62
10^6	469	690	1680	96	109	239	2.89
10^7	1485	2098	10080	336	356	574	3.66

$\#(n^2)$: number of additions to generate $1, 4, 9, \ldots, n^2$
$\#(\mathrm{mod}\ m)$: number of additions to generate $1, 4, 9, \ldots \bmod m$
$\#(\mathrm{tot})$: total multiplications in the rectangular splitting algorithm

The end